

ÉTALE COHOMOLOGICAL DIMENSION, A CONJECTURE OF LYUBEZNIK AND BOUNDS FOR ARITHMETIC RANK

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ABSTRACT. We produce a criterion for open sets in projective n -space over a separably closed field to have étale cohomological dimension bounded by $2n - 3$. We use the criterion to exhibit a scheme for which étale cohomological dimension is smaller than what a conjecture of G. Lyubeznik predicts; the discrepancy is of arithmetic nature.

For a monomial ideal, we relate extremal graded Betti numbers and étale cohomological dimension of the complement of the corresponding subspace arrangement. Moreover, we derive upper bounds for its arithmetic rank in terms of invariants distilled from the lcm-lattice.

1. INTRODUCTION

Let R be a Noetherian ring and I an R -ideal. The *arithmetic rank* of I , denoted $\text{ara } I$, is the least integer r for which there exist $g_1, \dots, g_r \in R$ such that $\sqrt{(g_1, \dots, g_r)} = \sqrt{I}$. Determining the exact value of the arithmetic rank of an ideal is in general a very difficult problem, and a complete answer is known only in a small set of cases. In a natural way, lower bounds for $\text{ara}(I)$ come out of non-vanishing results for suitable cohomology groups; on the other hand, finding upper bounds usually requires explicit constructions. The purpose of this paper is to provide some results of both kinds.

Let U be a Noetherian \mathbb{k} -scheme. The *quasi-coherent cohomological dimension* of U ,

$$(1.1) \quad \text{qccd}(U) := \max\{i : H^i(U, F) \neq 0, \text{ for all quasi-coherent sheaves } F \text{ on } U\},$$

is very closely connected to arithmetic rank. For example, suppose that X is an affine \mathbb{k} -scheme with coordinate ring R . Let Z be the subscheme of X defined by an R -ideal I and take $U = X \setminus Z$. Then

$$(1.2) \quad \text{ara } I \geq \text{qccd}(U) + 1.$$

Let $U_{\text{ét}}$ be the *small étale site* on U ; see [Mil80, Section II.1]. By a *torsion sheaf* on $U_{\text{ét}}$, we mean a sheaf of Abelian torsion groups on $U_{\text{ét}}$ on which $\text{char } \mathbb{k}$ operates as a unit. Let $\ell \neq \text{char } \mathbb{k}$ be a prime number; an ℓ -*torsion sheaf* on $U_{\text{ét}}$ is a torsion sheaf on which the action of ℓ is nilpotent. The *étale cohomological dimension* $\text{écd}(U)$ and the *étale ℓ -cohomological dimension* $\text{écd}_{\ell}(U)$ of U are measures of the topological complexity of the underlying variety [Lyu93, Sections 10 and 11]; they are defined by

$$(1.3) \quad \begin{aligned} \text{écd}(U) &:= \max\{i : H^i(U_{\text{ét}}, F) \neq 0, \text{ for all torsion sheaves } F \text{ on } U_{\text{ét}}\}, \text{ and,} \\ \text{écd}_{\ell}(U) &:= \max\{i : H^i(U_{\text{ét}}, F) \neq 0, \text{ for all } \ell\text{-torsion sheaves } F \text{ on } U_{\text{ét}}\}. \end{aligned}$$

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It is known that $\text{écd}(U) = \max\{\text{écd}_\ell(U) : \ell \neq \text{char } \mathbb{k} \text{ is a prime number}\}$, and this number gives another estimate for arithmetic rank:

$$(1.4) \quad \text{ara } I \geq \text{écd}(U) - \dim U + 1.$$

Consider an n -dimensional scheme U of finite-type over a separably closed field \mathbb{k} . The most general vanishing result is: $\text{écd}(U) \leq 2n$, [Mil80, Theorem VI.1.1]. There are results refining this bound, if U can be embedded in a nice ambient space X , and if some information about $X \setminus U$ is known. Such information, typically, includes $\dim(X \setminus U)$, the number of components and the combinatorial data on their intersections. For example, $\text{écd}(U) \leq 2n - 1$ if and only if no n -dimensional component of U is proper [Lyu93, Theorem 3.5]. One step further, for a non-singular affine variety X and a closed subscheme Z of X , [Lyu93, Theorem 4.9] shows that $\text{écd}(X \setminus Z) \leq 2n - 2$ if and only if every irreducible component of Z has dimension ≥ 2 and $Z \setminus \{p\}$ is locally analytically connected at p for every closed point $p \in Z$. Suppose now in addition that each component of Z has dimension at least 3. Under what circumstances can one hope to push the estimate for $\text{écd}(U)$ one step further? Topological information on the components of Z together with combinatorial data regarding how they are stacked together is not enough, see Remark 2.24.(c). On the positive side, we give in Theorem 2.2 a sufficient condition to ensure that $\text{écd}(U) \leq 2n - 3$, the additional ingredient 2.2.(c) being of arithmetic nature.

Both estimates (1.2) and (1.4) result from covering U by $\text{ara } I$ affine open subsets. The inequalities also hold when X is a projective \mathbb{k} -scheme, Z is a closed subscheme and $U = X \setminus Z$. For, if X' and Z' are the respective affine cones and $U' = X' \setminus Z'$, by [BS98, Chapter 20] and [Lyu93, Proposition 10.1] (also see (2.9) below),

$$(1.5) \quad \text{qccd}(U') = \text{qccd}(U) \text{ and } \text{écd}(U') = \text{écd}(U) + 1.$$

The following conjecture of G. Lyubeznik resulted from an attempt to understand whether one of the estimates of (1.2) and (1.4) provides a better bound than the other, in general:

Conjecture 1.6 ([Lyu02, p. 147]). Let U be a \mathbb{k} -scheme. Then $\text{écd}(U) \geq \text{qccd}(U) + \dim U$.

Conjecture 1.6 holds for (a) complements of certain determinantal varieties, if $\text{char } \mathbb{k} > 0$ [BS90, p.440], (b) complements of certain Segre embeddings, in characteristic zero [Ogu73, Example 4.6], and, (c) complements of smooth projective varieties, in characteristic zero [Var10, Theorem 2.21]. However, in Example 2.23, we combine bounds for étale cohomological dimension from Theorem 2.2 with arithmetic information from Theorem 2.19 and provide a counterexample to Conjecture 1.6.

The *cohomological dimension* of I on R is defined by:

$$(1.7) \quad \text{cd}(I, R) := \max\{i : H_i^I(M) \neq 0, \text{ for all } R\text{-modules } M\}.$$

Suppose that either $X = \text{Spec } R$ is affine or that R is graded, $X = \text{Proj } R$ and I is homogeneous. Let Z be defined by I . Then

$$(1.8) \quad \text{cd}(I, R) = \text{qccd}(X \setminus Z) + 1$$

In the affine case, see [ILL⁺07, Proposition 9.12]; the projective case follows from it, using (1.5).

Section 3 deals with obtaining upper bounds for arithmetic rank for monomial ideals. Let $R = \mathbb{k}[x_0, \dots, x_n]$; write \mathfrak{m} for the homogeneous maximal ideal of R . Let I be a homogeneous R -ideal. In this case, we may also consider the *homogeneous arithmetic rank* of I , which is the least integer r such that there exist homogeneous polynomials $g_1, \dots, g_r \in R$ such that $\sqrt{(g_1, \dots, g_r)} = \sqrt{I}$. We denote it by $\text{ara}_h I$. While $\text{ara } I$ gives the least number of equations to define $\text{Spec } R/I \subseteq \mathbb{A}^{n+1}$ set-theoretically, $\text{ara}_h I$ gives the least number of equations to define $\text{Proj } R/I \subseteq \mathbb{P}^n$ set-theoretically. Note that $\text{ara } I \leq \text{ara}_h I$; whether equality holds always is unknown.

Definition 1.9. We write $\text{Mon}(R)$ for the set of monomials of R . Suppose that I is a monomial R -ideal. We denote the set of the (unique) minimal monomial generators of I by $\text{Gen}(I)$. The *lcm-lattice* of I , denoted L_I , is the set of the least common multiples of the elements of $\text{Gen}(I)$, partially ordered by divisibility. The unique minimal element of L_I , which we denote by $\hat{0}$, is the monomial 1. Let L be any finite lattice. For $\tau, \sigma \in L$, let $[\tau, \sigma] := \{\rho : \tau \preceq \rho \preceq \sigma\}$ and $(\tau, \sigma) := \{\rho : \tau \prec \rho \prec \sigma\}$, both inheriting the order from L . For $\sigma \in L$, the *height* of σ , written $\text{ht}_L \sigma$, is the length of the longest (saturated) chain in $[\hat{0}, \sigma]$. The *height* of L , written $\text{ht } L$, is the length of the longest (saturated) chain in L . The elements of L of height 1 are called *atoms*.

The lcm-lattice L_I is an atomic lattice; the atoms of L_I are precisely the monomial generators of I . From L_I we construct homogeneous generators g_i of I up to radical. This yields two upper bounds. First, in Theorem 3.4 we show that $\text{ara}_h(I) - 1$ is bounded above by the dimension of any rooting complex (Definition 3.1) on L_I , generalizing a theorem of K. Kimura [Kim09, Theorem 1]. Later, in Proposition 3.7, we show that $\text{ara}_h(I) - 1$ is bounded above by the height of L_I .

2. ÉTALE COHOMOLOGICAL DIMENSION

We look first at étale cohomological dimensions of open subschemes of $\mathbb{P}_{\mathbb{k}}^n$. Then we express the étale cohomological dimension of complements of affine coordinate subspace arrangements in terms of extremal Betti numbers of their defining ideals. We combine these results to construct a counterexample to a conjecture of Lyubeznik.

Notation 2.1. Throughout this section, we take \mathbb{k} to be separably closed. Let $\ell \neq \text{char } \mathbb{k}$ be a prime number. We will abbreviate ‘a sheaf of \mathbb{Z}/ℓ -modules’ by ‘a \mathbb{Z}/ℓ -module’. We will omit the subscript ét in the proofs in this section, since all the cohomology groups in the proof are for (small) étale sites. For any scheme X and a point $p \in X$, write $\dim p = \dim \overline{\{p\}}$. For a real number r , $\lfloor r \rfloor$ denotes the unique integer such that $r - 1 < \lfloor r \rfloor \leq r$. For a local ring A , we write A^{sh} for its strict Henselization. If $j : U \rightarrow X$ is an étale morphism of finite type and F is a sheaf on $X_{\text{ét}}$, then we write $F|_U$ for j^*F [Mil80, Remark II.3.1.(a)].

Open subschemes of \mathbb{P}^n . We find an upper bound for the étale cohomological dimension of the complements of certain projective schemes. Call an irreducible subvariety Z of $\mathbb{P}_{\mathbb{k}}^n$, defined by a prime R -ideal I , *analytically irreducible at the vertex* if $\widehat{IR_{\mathfrak{m}}}$ (where $\widehat{}$ denotes completion) is a prime ideal.

Theorem 2.2. *Let $Z \subseteq \mathbb{P}_{\mathbb{k}}^n$ be a closed subscheme without any zero-dimensional components, defined by the ideal sheaf \mathcal{I} . Suppose that*

- (a) *In the stalk $\mathcal{O}_{\mathbb{P}^n, p}$, $\text{ara } \mathcal{I}_p \leq \min\{n - 2, n - \dim p - 1\}$ for all $p \in Z$ different from the generic points of Z ,*
- (b) *the irreducible components of Z are analytically irreducible at the vertex and have pairwise non-empty intersections, and,*
- (c) $H^{\geq 2n-2}((\mathbb{P}_{\mathbb{k}}^n \setminus Z)_{\text{ét}}, \mathbb{Z}/\ell) = 0$.

Then $\text{écd}_{\ell}(\mathbb{P}_{\mathbb{k}}^n \setminus Z) \leq 2n - 3$.

See Remark 2.25 below for a discussion of the hypotheses of Theorem 2.2. The key step in the proof of the theorem is Lemma 2.15, which establishes that certain local étale cohomology sheaves on the affine cones are not supported at the vertex. Using this result, we get a bound for the étale cohomological dimension of closed subschemes of $(\mathbb{P}_{\mathbb{k}}^n \setminus Z)$, in Lemma 2.16. We start by collecting some basic facts of étale cohomology.

Discussion 2.3. Let X be a scheme, and Z a closed subscheme of X ; denote the complementary open embedding by $j : (X \setminus Z) \rightarrow X$. We write $\mathcal{H}_Z^j(X, -)$ for the j th right derived functor of $F \mapsto \ker(F \rightarrow j_* j^* F)$. Let F be any \mathbb{Z}/ℓ -module on X . First, [Lyu93, Corollary 2.8] gives

$$(2.4) \quad \dim \operatorname{Supp} \mathcal{H}_Z^j(X, F) \leq \left\lfloor \dim \operatorname{Supp} F - \frac{j}{2} \right\rfloor, \quad \text{for all } j.$$

Secondly, let $p \in Z$ and write $A = (\mathcal{O}_{X,p})^{\text{sh}}$. Then, by [Lyu93, (1.11b)],

$$(2.5) \quad \mathcal{H}_Z^i(X, F)_p = H_Z^i(\operatorname{Spec}(\mathcal{O}_{X,p})^{\text{sh}}, F|_{\operatorname{Spec}(\mathcal{O}_{X,p})^{\text{sh}}}).$$

Further, we have a spectral sequence (see [Lyu93, (1.3a)])

$$(2.6) \quad E_2^{ij} = H^i(X, \mathcal{H}_Z^j(X, F)) \implies H_{Z'}^{i+j}(X, F).$$

If $X = \operatorname{Spec} A$, where A is a strictly Henselian ring, then we have $E_2^{ij} = 0$ for all $i > 0$, so,

$$(2.7) \quad H^0(\operatorname{Spec} A, \mathcal{H}_Z^j(\operatorname{Spec} A, F)) = H_Z^j(\operatorname{Spec} A, F), \quad \text{for all } j.$$

If, additionally, A is the strict Henselization of a ring that is essentially of finite type over a field, and Z is defined an ideal I , then (by [Lyu93, Proposition 2.9])

$$(2.8) \quad H_Z^j(\operatorname{Spec} A, F) = 0, \quad \text{for all } j > \dim A + \operatorname{ara} I, \text{ and for all } F.$$

Now suppose that X and Z are projective, and write X' and Z' for their respective affine cones. Then, by [Lyu93, Proposition 10.1],

$$(2.9) \quad H^{\geq t+1}(X' \setminus Z', \pi^* F) = 0 \text{ if and only if } H^{\geq t}(X \setminus Z, F) = 0$$

for every sheaf F on $(X \setminus Z)_{\text{ét}}$ and for every $t \geq 0$. \square

Notation 2.10. We write I for the defining ideal of Z and $\operatorname{Var}(I)$ for $\{\mathfrak{p} \in \operatorname{Spec} R : I \subseteq \mathfrak{p}\}$.

Lemma 2.11. *Assume hypothesis 2.2.(a). Then $\operatorname{ara} IR_{\mathfrak{p}} \leq n - 2$ for all $\mathfrak{p} \in \operatorname{Var}(I) \setminus \{\mathfrak{m}\}$. Moreover, $\operatorname{ara} IR_{\mathfrak{p}} \leq \operatorname{ht} \mathfrak{p} - 1$ for all $\mathfrak{p} \in \operatorname{Var}(I) \setminus (\{\mathfrak{m}\} \cup \operatorname{Min}(I))$.*

Proof. Take $\mathfrak{p} \neq \mathfrak{m} \in \operatorname{Var}(I)$ and write $\mathfrak{p}^* \neq \mathfrak{m}$ for the largest homogeneous R -subideal of I . We denote the corresponding point in Z by p . Since $\mathcal{O}_{\mathbb{P}^n, p} = \{\frac{f}{g} \mid g \notin \mathfrak{p}; f, g \text{ homogeneous of equal degree}\}$ (modulo the usual equivalence relation), there is a natural map $\mathcal{O}_{\mathbb{P}^n, p} \rightarrow R_{\mathfrak{p}}$ under which \mathcal{S}_p extends to $IR_{\mathfrak{p}}$. In particular,

$$(2.12) \quad \operatorname{ara}(IR_{\mathfrak{p}}) \leq \operatorname{ara}(\mathcal{S}_p) \quad \text{for all } \mathfrak{p} \in \operatorname{Var}(I) \setminus \{\mathfrak{m}\}.$$

If p is not a generic point of Z , then (2.12), together with hypothesis 2.2.(a), implies that $\operatorname{ara}(IR_{\mathfrak{p}}) \leq \min\{n - 2, n - \dim p - 1\} = \min\{n - 2, \operatorname{ht} \mathfrak{p}^* - 1\} \leq \min\{n - 2, \operatorname{ht} \mathfrak{p} - 1\}$.

So from now on assume $p \in Z$ to be generic (hence not closed), and let $p' \in \overline{\{p\}} \subseteq Z$ be a point of dimension $\dim p - 1$. Since $\mathcal{S}_p = \mathcal{S}_{p'} \mathcal{O}_p$, $\operatorname{ara}(\mathcal{S}_p) \leq \operatorname{ara}(\mathcal{S}_{p'})$. Applying (2.12) and hypothesis 2.2.(a), we find

$$\operatorname{ara}(IR_{\mathfrak{p}}) \leq \operatorname{ara}(\mathcal{S}_p) \leq \operatorname{ara}(\mathcal{S}_{p'}) \leq \min\{n - 2, n - \dim p' - 1\} \leq n - 2 \quad \text{for all } \mathfrak{p} \in \operatorname{Var}(I) \setminus \{\mathfrak{m}\}.$$

Finally, if $\mathfrak{p} \notin \operatorname{Min}(I)$ and p is generic, $\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}^*) + 1$ (see [Mat89, Exercise 13.6]). Thus, in this case we find $\operatorname{ara}(IR_{\mathfrak{p}}) \leq \min\{n - 2, n - \dim p' - 1\} = \min\{n - 2, \operatorname{ht} \mathfrak{p}^*\} = \min\{n - 2, \operatorname{ht} \mathfrak{p} - 1\}$. \square

Notation 2.13. Let $Y \subseteq \mathbb{P}_{\mathbb{k}}^n$ be an irreducible and reduced hypersurface such that $Z \subseteq Y$. Let $Z' \subseteq Y'$ (where $Z' = \operatorname{Spec} R/I$) be the affine cones over Z and Y , respectively. Write v for the vertex of Y' (and of Z'), and $(A, \mathfrak{m}_A) = (\mathcal{O}_{Y', v})^{\text{sh}}$.

Lemma 2.14. *Adopt Notation 2.13. The hypotheses 2.2.(a) and 2.2.(b) imply, respectively:*

- (a') $\text{ara } IA_{\mathfrak{q}} \leq \min\{n-2, \text{ht } \mathfrak{q} - 1\}$, for all $\mathfrak{q} \in \text{Var}(I) \setminus (\{\mathfrak{m}_A\} \cup \text{Min}(IA))$.
 (b') for all minimal primes \mathfrak{p} and \mathfrak{q} over IA , $\dim A/(\mathfrak{p} + \mathfrak{q}) > 0$.

Proof. Write $(S, \mathfrak{m}_S) = \mathcal{O}_{Y', v}$. (a'): Let $\mathfrak{q} \in \text{Var}(I) \setminus (\{\mathfrak{m}_A\} \cup \text{Min}(IA))$. Write $\bar{\mathfrak{q}} = \mathfrak{q} \cap S$. Since $A_{\mathfrak{q}}$ is a localization of $(S \setminus \bar{\mathfrak{q}})^{-1}A$, it suffices to show that $\text{ara } IS_{\bar{\mathfrak{q}}} \leq \min\{n-2, \text{ht } \mathfrak{q} - 1\}$. Lemma 2.11 gives $\text{ara } IS_{\bar{\mathfrak{q}}} \leq n-2$. By the going-down property (see [Mat89, 15.1]) of the flat extension $S \rightarrow A$, $\text{ht } \bar{\mathfrak{q}} \leq \text{ht } \mathfrak{q}$. Hence, if $\bar{\mathfrak{q}} \notin \text{Min}(IS)$, then (again by Lemma 2.11) $\text{ara } IS_{\bar{\mathfrak{q}}} \leq \text{ht } \mathfrak{q} - 1$. If $\bar{\mathfrak{q}} \in \text{Min}(IS)$, then $\text{ht } \bar{\mathfrak{q}} < \text{ht } \mathfrak{q}$, so, $\text{ara } IS_{\bar{\mathfrak{q}}} \leq \text{ht } \bar{\mathfrak{q}} \leq \text{ht } \mathfrak{q} - 1$.

(b'): Since \mathbb{k} is separably closed, A is, in fact, the Henselization of S . If \mathfrak{p} is minimal over IA , then $\mathfrak{p} = (\mathfrak{p} \cap S)A$, by 2.2.(b), since $\hat{A} = \hat{S}$. Hence, for \mathfrak{p} and \mathfrak{q} be minimal over IA ,

$$A/(\mathfrak{p} + \mathfrak{q}) = \left(\frac{S}{(\mathfrak{p} \cap S) + (\mathfrak{q} \cap S)} \right)^{\text{sh}},$$

which has positive dimension, again by 2.2.(b). (Note that taking the strict Henselization commutes with going modulo an ideal [Mil80, p. 38].) \square

Lemma 2.15. *Adopt Notation 2.13. Then, under the hypotheses 2.2.(a) and 2.2.(b),*

$$\mathcal{H}_{Z'}^{\geq 2n-1}(Y', F)_v = 0, \text{ for all } F.$$

Proof. In light of (2.5) and (2.7), it suffices to show that $\mathcal{H}_{Z'}^{\geq 2n-1}(\text{Spec } A, F) = 0$, which we will achieve using [Lyu93, Theorem 4.5]. Instead of using 2.2.(a) and 2.2.(b), we will use the corresponding statements from Lemma 2.14.

The embedding dimension of A is at most $n+1$. From Lemma 2.14.(a'), we see that every irreducible component of Z' has dimension at least 3, so $c(I) \leq n-2$, in the notation of [Lyu93, Theorem 4.5]. We want to verify its hypotheses, taking $t = 2n-1$.

(i): the condition can be rephrased as follows: for all j , $\dim \text{Supp } \mathcal{H}_{Z'}^j(\text{Spec } A, F) \leq \left\lfloor \frac{t-j+1}{2} \right\rfloor$, if $t-j \leq 2c(I)-1$. This, in turn, follows from (2.4), since $\dim \text{Supp } F \leq \dim A = n = \frac{t+1}{2}$.

(ii): Let s be an odd integer such that $1 \leq s \leq 2c(I)-1$. If $j \geq t-s+1$, then, by (2.4), we see that $\dim \text{Supp } \mathcal{H}_{Z'}^j(\text{Spec } A, F) \leq \left\lfloor \frac{s}{2} \right\rfloor = \frac{s+1}{2} - 1$, so $\mathcal{H}_{Z'}^j(\text{Spec } A, F)$ is not supported at geometric points of dimension $\frac{s+1}{2}$. Hence assume that $j = t-s$. Let $q \in Z'$ be such that $\dim q = \frac{s+1}{2}$, defined by a prime A -ideal \mathfrak{q} such that $A/(\mathfrak{q} + \mathfrak{q}') = 0$ for some prime A -ideal \mathfrak{q}' that is minimal over IA . We need to show that $\mathcal{H}_{Z'}^{t-s}(\text{Spec } A, F)_q = 0$. It follows from Lemma 2.14.(b') that \mathfrak{q} is not a minimal prime over IA . Write $B = (A_{\mathfrak{q}})^{\text{sh}}$. By Lemma 2.14.(a'), $t-s = (2n-1) - (2 \dim q - 1) = 2 \dim B > \dim B + \text{ara } IB$. By (2.5), and (2.8) $\mathcal{H}_{Z'}^{t-s}(\text{Spec } A, F)_q = H_{Z'}^{t-s}(\text{Spec } B, F|_{\text{Spec } B}) = 0$.

(iii): Note that it suffices to check for $j = t-s$ ($q = t-s$, in the notation of [Lyu93, Theorem 4.5].) Now use (2.4), and the fact that for all $s \geq 2c(I)$, $s - c(I) \geq \left\lfloor \frac{s+1}{2} \right\rfloor = \left\lfloor n - \frac{t-s}{2} \right\rfloor$. \square

Lemma 2.16. *Under the hypotheses 2.2.(a) and 2.2.(b), $\text{écd}_{\ell}(Y \setminus Z) \leq 2n-4$ for all closed subschemes $Y \subsetneq \mathbb{P}_{\mathbb{k}}^n$.*

Proof. We may assume that Y is an irreducible and reduced hypersurface that contains Z (see [Mil80, VI.1.1], and the first paragraph of its proof) and that Z is reduced. Now adopt Notation 2.13.

Let G be any \mathbb{Z}/ℓ -module on $(Y \setminus Z)$. We need to show that $H^{\geq 2n-3}((Y \setminus Z), G) = 0$. By (2.9) it suffices to show that $H^{\geq 2n-2}(Y' \setminus Z', \pi^*G) = 0$. Let F be a sheaf of \mathbb{Z}/ℓ -modules on Y' such that $F|_{(Y' \setminus Z')} = \pi^*G$. Since Y' is affine, $H^{\geq 2n-2}(Y' \setminus Z', \pi^*G) = 0$ if and only if $H_{Z'}^{\geq 2n-1}(Y', F) = 0$; see [Mil80, III.1.25].

We obtain, from Lemma 2.11, that if $p \in Z'$ and $p \neq v$ then $\text{ara } I(\mathcal{O}_{Y',p})^{\text{sh}} \leq n - 2$. Therefore, by (2.8), for all $p \neq v \in Y'$, $\mathcal{H}_{Z'}^j(Y', F)_p = 0$ for all $j > 2n - \dim p - 2$. In particular, the stalks $\mathcal{H}_{Z'}^{\geq 2n-1}(Y', F)_p$ are zero, for all $p \neq v \in Y'$. Combining this with Lemma 2.15 and (2.4), we see that

$$\mathcal{H}_{Z'}^{\geq 2n-1}(Y', F) = 0 \text{ and } \dim \text{Supp } \mathcal{H}_{Z'}^j(Y', F) \leq \min \left\{ \left\lfloor n - \frac{j}{2} \right\rfloor, 2n - j - 2 \right\}, \text{ for all } j \leq 2n - 2.$$

Since Y' is affine, $H^i(Y', \mathcal{H}_{Z'}^j(Y', F)) = 0$, for all $i > \dim \text{Supp } \mathcal{H}_{Z'}^j(Y', F)$. Using (2.6), we conclude that $H_{Z'}^{\geq 2n-1}(Y', F) = 0$. \square

Discussion 2.17. In order to determine étale cohomological dimension, it suffices to consider constructible sheaves. Suppose that X is Noetherian. A sheaf F (of \mathbb{Z}/ℓ -modules, for us) on X is *constructible* if every irreducible closed subscheme Z of X contains a nonempty open subscheme U such that $F|_U$ is locally constant and has finite stalks [Mil80, V.1.8(b)]. Let F be a constructible \mathbb{Z}/ℓ -module on $X_{\text{ét}}$. Then F has a finite filtration by subsheaves such that successive quotients are of the form $j_!G$ where $j : U \rightarrow X$ is the inclusion of an irreducible, locally closed and constructible subset, and G is a locally constant and constructible \mathbb{Z}/ℓ -module on U ; see [SGA4, IX.2.5], using [SGA4, IX.2.4.(i)] and [Mil80, V.1.8(b)]. \square

Proof of Theorem 2.2. Write $U = \mathbb{P}^n \setminus Z$ and $r = \text{écd}_{\ell}(U)$. We want to show that $r \leq 2n - 3$. Let F be a sheaf of \mathbb{Z}/ℓ -modules on U such that $H^r(U, F) \neq 0$. We may assume that F is constructible. By induction on the length of the filtration in Discussion 2.3, we may assume that $F = j_!G$ where $j : V \rightarrow U$ is the inclusion of an irreducible locally closed subset, and G is a locally constant and constructible \mathbb{Z}/ℓ -module on V . If j is not an open immersion, then either $\dim \text{Supp } F \leq n - 2$ or $\dim \text{Supp } F = n - 1$ and (since $\dim Z \geq 2$ by 2.2.(a)) every maximum-dimensional component of $\text{Supp } F$ is non-proper. In either case, $\text{écd}(U) \leq 2n - 3$. (The latter case follows from [Lyu93, Theorem 3.5].) Therefore we may assume that j is an open immersion.

Write η for the geometric generic point of U and V , and let $i : \eta \rightarrow V$ be the inclusion map. Write $F_1 = \ker(F \rightarrow i_*i^*F)$ and $F_2 = \text{coker}(F \rightarrow i_*i^*F)$, and consider the exact sequence

$$0 \rightarrow j_!F_1 \rightarrow j_!F \rightarrow j_!i_*i^*F \rightarrow j_!F_2 \rightarrow 0.$$

on U . (Note that $j_!$ is exact [Mil80, II.3.1.4].) Since the stalks $(j_!F)_{\eta}$ and $(j_!i_*i^*F)_{\eta}$ are identical, $j_!F_1$ and $j_!F_2$ are supported in dimension at most $n - 1$. By Proposition 2.16, $H^{\geq 2n-3}(U, j_!F_1) = H^{\geq 2n-3}(U, j_!F_2) = 0$. Hence, to show that $r \leq 2n - 3$, it suffices to show that $H^{\geq 2n-2}(U, j_!i_*i^*F) = 0$. Observe that i_*i^*F is a constant sheaf on V . Hence, without loss of generality, $F = j_!(\mathbb{Z}/\ell)$.

Let ι be the inclusion map $\eta \rightarrow U$, $F_1 = \ker(F \rightarrow \iota_*\iota^*F)$ and $F_2 = \text{coker}(F \rightarrow \iota_*\iota^*F)$. Consider the exact sequence

$$0 \rightarrow F_1 \rightarrow F \rightarrow \iota_*\iota^*F \rightarrow F_2 \rightarrow 0.$$

Note that $\iota_*\iota^*F$ is a constant \mathbb{Z}/ℓ -module on U and that F_1 and F_2 are supported in dimension at most $n - 1$. By Lemma 2.16, $H^{\geq 2n-3}(U, F_1) = H^{\geq 2n-3}(U, F_2) = 0$, while by hypothesis 2.2.(c), $H^{\geq 2n-2}(U, \iota_*\iota^*F) = 0$. Hence $H^{\geq 2n-2}(U, F) = 0$. \square

Affine coordinate subspace arrangements. We now turn to the complements of affine coordinate subspace arrangements. We will relate their étale cohomology groups to graded Betti numbers of the ideals defining the arrangements. Then the étale cohomological dimension can be expressed in terms of extremal Betti numbers.

Notation 2.18. Let $S = \mathbb{Z}[x_0, \dots, x_n]$ and let \mathfrak{a} be a square-free monomial S -ideal. Let $R = \mathbb{k}[x_0, \dots, x_n]$ (as usual) and $R' = \mathbb{Z}/\ell[x_0, \dots, x_n]$. Let $I = \mathfrak{a}R$ and $J = \mathfrak{a}R'$. Let Z be the closed \mathbb{k} -subscheme of $\mathbb{A}_{\mathbb{k}}^{n+1}$ defined by I .

Theorem 2.19. *With the notation as above,*

$$\max\{r : H^r((\mathbb{A}_{\mathbb{k}}^{n+1} \setminus Z)_{\text{ét}}, \mathbb{Z}/\ell) \neq 0\} = \max\{i + j : \beta_{i,j}^{R'}(J) \neq 0\}$$

Notation 2.20. Let \mathcal{A} be the coordinate subspace arrangement defined by I inside \mathbb{k}^{n+1} . Let $(L_{\mathcal{A}}, \preceq)$ be the intersection lattice of \mathcal{A} , partially ordered by *inclusion*: $v \preceq w$ if $v \subseteq w$. By $(-)^{\vee}$, we denote the operation of taking the Alexander dual of a square-free monomial ideal. Let $\mu : L_{\mathcal{A}} \rightarrow L_{I^{\vee}}$ be a map of posets with $\mu(v) = \prod_{x_i(v)=0} x_i$. (See Definition 1.9 for the lcm-lattice $L_{I^{\vee}}$ of I^{\vee} .) Let (P, \preceq) be a poset. Write $\Delta(P)$ for its order complex, *i.e.*, the simplicial complex on P with faces $\{v_1, \dots, v_r\}$ whenever $v_1 \preceq v_2 \preceq \dots \preceq v_r$. For $v \in P$, write $P_{\succeq v}$ for the induced poset on $\{w \in P : v \preceq w\}$, and $P_{\succ v}$ for the induced poset on $\{w \in P : v \not\preceq w\}$. Write $U = \mathbb{A}_{\mathbb{k}}^{n+1} \setminus Z$.

The following was, perhaps, first observed by V. Gasharov, I. Peeva and V. Welker [GPW02].

Lemma 2.21. *The map μ defines a duality between $L_{\mathcal{A}}$ and $L_{I^{\vee}}$.*

Sketch. (See, also, [GPW02, Proof of Theorem 3.1].) The coatoms of $L_{\mathcal{A}}$ correspond to the irreducible components of Z , which correspond, under μ , to the minimal generators of I^{\vee} , *i.e.*, the atoms of $L_{I^{\vee}}$. The flats of $L_{\mathcal{A}}$ are intersections of the coatoms; this corresponds to taking the lcm of the various subsets of $\text{Gen}(I)$. \square

Definition 2.22 ([BCP99, p. 498]). A Betti number $\beta_{i,j}(M) \neq 0$ is *extremal* if $\beta_{k,l}(M) = 0$ for all (k, l) such that $(k, l) \neq (i, j)$, $k \geq i$ and $j - i \geq l - k$.

Proof of Theorem 2.19. Write $L = L_{\mathcal{A}}$ and $L' = L_{I^{\vee}}$. For $v \in L$, write $d(v) = \dim_{\mathbb{k}} v$. Then,

$$H^r(U_{\text{ét}}, \mathbb{Z}/\ell) = \bigoplus_{v \in L} H^{2n-r-2d(v)-1}(\Delta(L_{\succeq v}), \Delta(L_{\succ v}); \mathbb{Z}/\ell).$$

This is [Yan00, Corollary 2, p. 311], taken along with the discussion in [Yan00, Section 2, pp. 306-7]. From the long exact sequence (we suppress the group \mathbb{Z}/ℓ of coefficients)

$$\cdots \rightarrow H^r(\Delta(L_{\succeq v}), \Delta(L_{\succ v})) \rightarrow H^r(\Delta(L_{\succeq v})) \rightarrow H^r(\Delta(L_{\succ v})) \rightarrow H^{r+1}(\Delta(L_{\succeq v}), \Delta(L_{\succ v})) \rightarrow \cdots$$

and the fact that $\Delta(L_{\succeq v})$ is a cone, we obtain the following:

$$H^{r+1}(\Delta(L_{\succeq v}), \Delta(L_{\succ v})) \simeq \tilde{H}^r(\Delta(L_{\succ v})), \text{ for all } r \geq 0.$$

From Lemma 2.21 we see that $\Delta(L_{\succ v})$ and $\Delta((\hat{0}, \mu(v))_{L'})$ are isomorphic. Therefore, by [GPW99, Theorem 2.1], we see that

$$\dim_{\mathbb{k}} H^r(U_{\text{ét}}, \mathbb{Z}/\ell) = \sum_{v \in L} \beta_{2n-r-2d(v), \deg \mu(v)}^{R'}(R'/J^{\vee}).$$

Write $t = \max\{r : H^r((\mathbb{A}_{\mathbb{k}}^{n+1} \setminus Z)_{\text{ét}}, \mathbb{Z}/\ell) \neq 0\}$. Note that $n - d(v) = \deg \mu(v)$. Hence

$$\begin{aligned} t &= \max\{r : \text{there exists } m \in L' \text{ such that } \beta_{2 \deg m - r, \deg m}^{R'}(R'/J^{\vee}) \neq 0\} \\ &= \max\{2j - i : \beta_{i,j}^{R'}(R'/J^{\vee}) \neq 0\} \\ &= \max\{2j - i : \beta_{i,j}^{R'}(R'/J^{\vee}) \text{ is an extremal Betti number of } (R'/J^{\vee})\}. \end{aligned}$$

(We see the last equality as follows: Pick i, j such that $\beta_{i,j}^{R'}(R'/J^{\vee}) \neq 0$ and $2j - i$ is maximized. If $\beta_{i,j}^{R'}(R'/J^{\vee})$ were not an extremal Betti number, then there would be i', j' such that

(a) $\beta_{i',j'}^{R'}(R'/J^\vee) \neq 0$, (b) $i' \geq i$ and $j' - i' \geq j - i$, and, (c) at least one of inequalities in (b) is strict. Then $2j' - i' > 2j - i$, a contradiction.)

Now, [BCP99, Theorem 2.8] gives the following: if $\beta_{i,j}^{R'}(R'/J^\vee)$ is an extremal Betti number of R'/J^\vee , then $\beta_{j-i,j}^{R'}(J) = \beta_{i,j}^{R'}(R'/J^\vee)$, and $\beta_{j-i,j}^{R'}(J)$ is an extremal Betti number of J . Therefore

$$\begin{aligned} t &= \max\{j + i : \beta_{i,j}^{R'}(R'/J^\vee) \text{ is an extremal Betti number of } (R'/J^\vee)\} \\ &= \max\{j + i : \beta_{i,j}^{R'}(R'/J^\vee) \neq 0\}. \end{aligned}$$

This proves the theorem. \square

In [GPW02, Theorem 3.1], Gasharov, Peeva and Welker prove an analogue of the theorem for the singular cohomology groups of affine real subspace arrangements.

Reisner's ideal and the conjecture of Lyubeznik. We present a counterexample to Conjecture 1.6 which asserts that for any \mathbb{k} -scheme U , $\text{écd}(U) \geq \dim U + \text{qccd}(U)$. We were unable to find such an example in the literature.

Example 2.23. Let $\text{char } \mathbb{k} = 2$. Let $R = \mathbb{k}[x_0, \dots, x_5]$, I the monomial R -ideal of the minimal triangulation of \mathbb{RP}^2 given by G. Reisner; see, e.g., [BH93, Section 5.3]. Then $\text{ht } I = 3$, $\text{pd } R/I = 4$ and $\text{ara } I = 4$ [Yan00, Example 2]. Let Z be the closed \mathbb{k} -scheme of $\mathbb{P}_{\mathbb{k}}^5$ defined by I and set $U = \mathbb{P}_{\mathbb{k}}^5 \setminus Z$. Write \mathcal{I} for the ideal sheaf of Z . Note that Z is a 2-dimensional projective scheme.

Since $\text{pd}_R R/I = 4$, $\text{cd}(I, R) = 4$ by [Lyu84, Theorem 1]. Thus, by (1.8), $\text{qccd}(U) = 3$. Hence $\dim U + \text{qccd}(U) = 8$. We apply Theorem 2.2 to show that $\text{écd}_\ell(U) \leq 7$ for all prime numbers $\ell \neq 2$. Let $\ell \neq 2$ be a prime number.

To show that the hypothesis 2.2.(a) holds, it suffices to show that the stalk \mathcal{I}_p is a set-theoretic complete intersection (i.e., $\text{ara } \mathcal{I}_p = \text{ht } \mathcal{I}_p$) for all $p \in Z$. Let $p \in Z$; without loss of generality, $x_0 \neq 0$ at p . We will show that IR_{x_0} is a set-theoretic complete intersection. This follows from noting that $(I :_R x_0) = \sqrt{(x_1x_2, x_5x_1 + x_3x_4, x_4x_5 + x_2x_3)}$ (see, e.g., [Bar08, p. 4545]). For hypothesis 2.2.(b), note that the irreducible components are linear subspaces of $\mathbb{P}_{\mathbb{k}}^5$ (so they are analytically irreducible) and that they have pairwise non-empty intersection (for instance, consider the triangulation given in [BH93, Section 5.3]). To see that hypothesis 2.2.(c) is satisfied, let R' and J be as in Notation 2.18. Write Z' for the affine cone over Z . Since J is a height three Cohen–Macaulay ideal generated by cubics and has a linear resolution, it has a single extremal Betti number $\beta_{2,5}(J)$, so, by Theorem 2.19, $H^7((\mathbb{A}_{\mathbb{k}}^6 \setminus Z')_{\text{ét}}, \mathbb{Z}/\ell) \neq 0 = H^{\geq 8}((\mathbb{A}_{\mathbb{k}}^6 \setminus Z')_{\text{ét}}, \mathbb{Z}/\ell)$. By (2.9), $H^6(U_{\text{ét}}, \mathbb{Z}/\ell) \neq 0 = H^{\geq 7}(U_{\text{ét}}, \mathbb{Z}/\ell)$. Now, by Theorem 2.2, $\text{écd}_\ell(U) \leq 7$. \square

Remark 2.24. (a) Theorems 5.3.(ii), and 6.3.(ii) of [Lyu93] reach the conclusion of Lemma 2.15, but with stronger hypotheses, which do not apply to Example 2.23.

(b) On Example 2.23: In view of the fact that $H^7(U_{\text{ét}}, \mathbb{Z}/\ell) = 0$ one might wonder whether $\text{écd}(U_{\text{ét}})$ equals 6. However, let Z_1 be the union of all the 2-dimensional coordinate subspaces of $\mathbb{P}_{\mathbb{k}}^5$; hence Z_1 is defined by $\bigcap_{0 \leq i < j < k \leq 5} (x_i, x_j, x_k)$. Theorem 2.19, together with (2.9), implies that $H^7((\mathbb{P}_{\mathbb{k}}^5 \setminus Z_1), \mathbb{Z}/\ell) \neq 0 = H^{\geq 8}((\mathbb{P}_{\mathbb{k}}^5 \setminus Z_1), \mathbb{Z}/\ell)$. Since $Z \subseteq Z_1$, we see that $\text{écd}_\ell(U) = 7$.

(c) More on Example 2.23: Instead of taking $\text{char } \mathbb{k} = 2$, we let $\text{char } \mathbb{k} = p > 0$. (As usual, $\ell \neq p$ is a prime number.) Let $Z \subseteq \mathbb{P}_{\mathbb{k}}^5$ and $Z' \subseteq \mathbb{A}_{\mathbb{k}}^6$ be defined by the monomial R -ideal I of the minimal triangulation of \mathbb{RP}^2 . Let J be as in Notation 2.18. The extremal Betti numbers of J are

$$\begin{aligned} &\beta_{2,6}(J) \text{ and } \beta_{3,6}(J), & \text{if } \ell = 2; \\ &\beta_{2,5}(J), & \text{if } \ell > 2. \end{aligned}$$

Suppose that $p = 2$ (and $\ell > 2$). Then

(i) $H^7(\mathbb{A}^6 \setminus Z', \mathbb{Z}/\ell) \neq 0 = H^{\geq 8}(\mathbb{A}^6 \setminus Z', \mathbb{Z}/\ell)$ and $H^6(\mathbb{P}^5 \setminus Z, \mathbb{Z}/\ell) \neq 0 = H^{\geq 7}(\mathbb{P}^5 \setminus Z, \mathbb{Z}/\ell)$. (Example 2.23.)

(ii) $\text{écd}(\mathbb{A}^6 \setminus Z') = \text{écd}_\ell(\mathbb{A}^6 \setminus Z') = 8$ and $\text{écd}(\mathbb{P}^5 \setminus Z) = \text{écd}_\ell(\mathbb{P}^5 \setminus Z) = 7$. (See (b) above.)

On the other hand, if $p > 2$ then

(i') $H^9(\mathbb{A}^6 \setminus Z', \mathbb{Z}/2) \neq 0 = H^{\geq 10}(\mathbb{A}^6 \setminus Z', \mathbb{Z}/2)$ and $H^8(\mathbb{P}^5 \setminus Z, \mathbb{Z}/2) \neq 0 = H^{\geq 9}(\mathbb{P}^5 \setminus Z, \mathbb{Z}/2)$. (Theorem 2.19 gives the affine case; the projective case then follows from (2.9).)

(ii') $\text{écd}(\mathbb{A}^6 \setminus Z') = \text{écd}_2(\mathbb{A}^6 \setminus Z') = 9$ and $\text{écd}(\mathbb{P}^5 \setminus Z) = \text{écd}_2(\mathbb{P}^5 \setminus Z) = 8$. (The projective case follows from Theorem 2.2, and it gives the affine case by (1.5).)

In particular, this example shows that topological data on closed subschemes and combinatorial data on their intersections do not suffice to predict the étale cohomological dimension of the complement of their union. \square

Remark 2.25. Hypothesis 2.2.(a) is crucial to the proof. Even with it, the other two hypotheses are independent of each other. In Remark 2.24.(c), we saw an example (with $p > 2$ and $\ell = 2$) in which the hypothesis 2.2.(b) holds, but the hypothesis 2.2.(c) does not.

The following is an example where 2.2.(c) is satisfied, but 2.2.(b) is not. Let $Z \subseteq \mathbb{P}_{\mathbb{k}}^5$ be defined by $I = (x_0x_1, x_2x_3, x_4x_5, x_0x_3, x_0x_5, x_2x_5)$, and let Z' be the affine cone over Z . (There are no restrictions on $\text{char } \mathbb{k}$ and ℓ , except that $\ell \neq \text{char } \mathbb{k}$.) Then Z is a codim 3 set-theoretic complete intersection (see, e.g., [Kum09, Theorem 1.3]). Its two irreducible components defined by (x_0, x_2, x_4) and (x_1, x_3, x_5) are disjoint. The ideal I has a linear resolution, so it has one extremal Betti number $\beta_{2,4}(I)$. Hence $H^{\geq 7}((\mathbb{A}^6 \setminus Z')_{\text{ét}}, \mathbb{Z}/\ell) = 0$, so $H^{\geq 6}((\mathbb{P}^5 \setminus Z)_{\text{ét}}, \mathbb{Z}/\ell) = 0$. \square

3. UPPER BOUNDS FROM THE LCM-LATTICE

Let L be the lcm-lattice of I . Using rooting complexes (see [BZ91, Section 3] and [Nov02]) we obtain an upper bound for $\text{ara}_h I$. Write L_1 for the set of atoms (i.e., elements of height one) of L ; this is the set of monomial minimal generators of I . For $F \subseteq L_1$, let $\lambda_F = \text{lcm } F$. For a simplicial complex Γ , denote the set of r -dimensional faces by $\Gamma^{(r)}$.

Definition 3.1. A *rooting map* is a function $\rho : L \rightarrow L_1$ such that for every $m \in L$, $\rho(m) | m$ and for all $m' \in [\rho(m), m]$, $\rho(m') = \rho(m)$. We say that $G \subseteq L_1$ is *unbroken* if $\rho(\lambda_G) \in G$. The *rooting complex* of L corresponding to ρ is $\Gamma_{I,\rho} = \{F \subseteq L_1 : G \text{ is unbroken, for all } G \subseteq F\}$.

Lemma 3.2 ([BZ91, Theorem 3.2(2)], [Nov02, Proposition 1(1)]). *The simplicial complex $\Gamma_{I,\rho}$ is a cone with apex $\rho(\lambda_{L_1})$.*

Lemma 3.3. *Let $I' = (I \cap \mathbb{k}[x_1, \dots, \widehat{x_r}, \dots, x_n])R$. Let L' be the lcm-lattice of I' , L'_1 , the set of atoms of L' , and $\rho' := \rho|_{L'}$. Then (a) ρ' is a rooting map of L' , and (b) $\Gamma_{I',\rho'} = \Gamma_{I,\rho}|_{L'_1}$.*

Proof. The proof of (a) is immediate. To prove (b), we need to show that for all $F \subseteq L'_1$, F is a face of $\Gamma_{I',\rho'}$ if and only if F is a face of $\Gamma_{I,\rho}$. Let $F \subseteq L'_1$. Then F is a face of $\Gamma_{I',\rho'}$ if and only if $\rho'(\lambda_G) \in G$ for all $G \subseteq F$, which holds if and only if $\rho(\lambda_G) \in G$ for all $G \subseteq F$, which holds if and only if F is a face of $\Gamma_{I,\rho}$. \square

The following theorem generalizes a result of K. Kimura [Kim09, Theorem 1], which shows that $\text{ara } I$ is at most the length of any Lyubeznik resolution (see [Lyu88]). I. Novik [Nov02, Remark after Theorem 1] showed that Lyubeznik resolutions arise as special cases of resolutions supported on rooting complexes. More precisely, rooting maps induce total orders on each rooted set; if these total orders are compatible with each other and induce a total order on $\text{Gen}(I)$, then the corresponding free resolution is a Lyubeznik resolution.

Theorem 3.4. Fix an integer $d \geq \max\{\deg f : f \in \text{Gen}(I)\}$. Let $\overline{(-)} : \text{Gen}(I) \rightarrow \text{Mon}(R)$ be a function such that $\deg \overline{f} = d$ and $\sqrt{\overline{f}} = \sqrt{f}$, for all $f \in \text{Gen}(I)$. For $r = 0, \dots, \dim \Gamma_{I,\rho}$, let

$$g_r = \sum_{F \in (\Gamma_{I,\rho})^{(r)}} \prod_{f \in F} \overline{f}.$$

Then $I = \sqrt{(g_i : 0 \leq r \leq \dim \Gamma_{I,\rho})}$. In particular, $\text{ara}_h I \leq 1 + \dim \Gamma_{I,\rho}$.

Proof. The second assertion follows from the first, after noting that for each r , g_r is a homogeneous polynomial of degree $(r+1)d$. We prove the first assertion by induction on $n = \dim R$. We may assume that it holds for all monomial ideals generated in $n-1$ or fewer variables.

Write $\Gamma = \Gamma_{I,\rho}$, $l = \dim \Gamma_{I,\rho}$ and $J = (g_0, \dots, g_l)$. For $F \subseteq L_1$, write $\pi_F = \prod_{f \in F} \overline{f}$. Without loss of generality, $\rho(\lambda_{L_1}) = f_1$. It follows from Lemma 3.2 that

$$(3.5) \quad g_r = \begin{cases} \overline{f_1} + \dots + \overline{f_m}, & r = 0 \\ \overline{f_1} \left(\sum_{\substack{F \in \Gamma^{(r-1)} \\ f_1 \notin F}} \pi_F \right) + \sum_{\substack{F \in \Gamma^{(r)} \\ f_1 \notin F}} \pi_F, & 1 \leq r \leq l-1 \\ \overline{f_1} \left(\sum_{\substack{F \in \Gamma^{(l-1)} \\ f_1 \notin F}} \pi_F \right), & r = l. \end{cases}$$

Let $\mathfrak{p} \in \text{Spec } R$ be such that $J \subseteq \mathfrak{p}$. We first claim that $\overline{f_1} \in \mathfrak{p}$. For, otherwise, it follows from (3.5) that $\overline{f_2} + \dots + \overline{f_r} \in \mathfrak{p}$, which implies that $\overline{f_1} \in \mathfrak{p}$, a contradiction; hence $\overline{f_1} \in \mathfrak{p}$. Therefore there exists i such that $x_i \in \mathfrak{p}$. Then $(J, x_i) \subseteq \mathfrak{p}$. Let $I' = (I \cap \mathbb{k}[x_1, \dots, \widehat{x_i}, \dots, x_n])R$. Let L' be the lcm-lattice of I' , L'_1 the set of atoms of L' , and $\rho' := \rho|_{L'}$. Since

$$g_r \equiv \sum_{F \in (\Gamma_{I',\rho'})^{(r)}} \prod_{f \in F} \overline{f} \pmod{x_i},$$

we see, by induction, that $\sqrt{(J, x_i)} = (I', x_i)$. Therefore $I \subseteq \mathfrak{p}$. \square

Since Lyubeznik resolutions arise as special cases of resolutions supported on rooting complexes, one might wonder whether it is possible to construct shorter resolutions by taking rooting maps that do not induce a global ordering of $\text{Gen}(I)$. This is the motivation of the following question. We, however, do not know the answer.

Question 3.6. Let L be any finite atomic lattice. Is

$$\min\{\dim \Gamma_{L,\rho} : \rho \text{ a rooting map}\} \geq \min\{\dim \Gamma_{L,\rho} : \rho \text{ a rooting map induced by a total order on } L_1\}?$$

The polynomials g_r in the previous theorem may be of relatively large degree. The following proposition provides another set of generators up to radical, of smaller degree, at the expense of increasing the number of generators.

Proposition 3.7. For $r = 1, \dots, \text{ht } L$, fix an integer $d_r \geq \max\{\deg \sigma : \sigma \in L, \text{ht } \sigma = r\}$. Let $\overline{(-)} : L \rightarrow \text{Mon}(R)$ be a function such that $\deg \overline{\sigma} = d_{\text{ht } \sigma}$ and $\sqrt{\overline{\sigma}} = \sqrt{\sigma}$, for all $\sigma \in L$. For $r = 1, \dots, \text{ht } L$, let

$$g_r = \sum_{\substack{\sigma \in L \\ \text{ht } \sigma = r}} \overline{\sigma}.$$

Then $I = \sqrt{(g_i : 1 \leq r \leq \text{ht } L)}$. In particular, $\text{ara}_h I \leq 1 + \text{ht } L$.

Proof. It suffices to show that $I \subseteq \sqrt{(g_i : 1 \leq r \leq \text{ht } L)}$, which we do by induction on $n = \dim R$. We may assume that the assertion holds for all monomial ideals generated in $n-1$ or fewer variables.

Write $l = \text{ht } L$ and $J = (g_1, \dots, g_l)$. Note that $g_l = x_1 x_2 \cdots x_n$. Let $\mathfrak{p} \in \text{Spec } R$ be such that $J \subseteq \mathfrak{p}$. Then, there exists t such that $(J, x_t) \subseteq \mathfrak{p}$. Let I' be the subideal of I generated by the minimal generators of I that are not divisible by x_t . Write L' for the lcm-lattice of I' . It is a sublattice of L , and, furthermore, for any $\sigma \in L'$, $\text{ht}_L \sigma = \text{ht}_{L'} \sigma$. Hence, by induction, $\sqrt{(J, x_t)} = (I', x_t) = (I, x_t)$. Therefore $I \subseteq \mathfrak{p}$. \square

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